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ANALYSIS OF THE SPATIAL NON-LINEAR VIBRATIONS OF A STRING†

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The problem of the free and forced quasilinear spatial vibrations of a string is investigated in the single-mode approximation of wave processes. A mathematical model is constructed in which geometric non-linearity, caused by the linear extensibility of a string, is the major source of non-linearity. As a result, the tension turns out to be variable both with respect to time and length. An asymptotic analysis of the free vibrations is carried out and the phenomenon of the instability of plane vibrations is investigated. The resonance curves corresponding to a plane harmonic excitation are constructed in terms of the system parameters and analysed. The stability of steady vibrations is completely investigated using Lyapunov's first method. Qualitative effects, associated with the stability and instability of plane and spatial forced vibrations, are detected and studied within the framework of the spatial model.

Experiments on free and forced vibrations of a string clamped at both ends demonstrate phenomena that are of a non-linear nature and cannot be explained using a linear model. The extensibility of the string material, which is neglected in the standard formulation of the problem on the transverse vibrations of a string [1–3], is one of the principal sources of non-linear effects. Furthermore, the tension in a string is usually assumed to be constant at all points [4–6, 8] (Kirchhoff's hypothesis) and in time [1–3].

According to a number of preliminary theoretical and experimental results which take account of non-linear effects [4–8], the spatial pattern of free and forced vibrations turns out to be rather complex. It requires the application of rigorous methods of non-linear mechanics to construct adequate mathematical models, to build systems of vibrations and to analyse their stability. In the single-mode quasilinear treatment of wave processes in a string which is considered below, the use of small-parameter methods (Lyapunov–Poincaré [9] and averaging [10, 11]) proves to be quite effective.

1. We shall construct a mathematical model of the non-linear vibrations of a string taking account of the variability of its length due to the extensibility of the string and the variability of the tension [4–7]. We introduce an inertial system of Cartesian coordinates xyz and assume that, when there are no transverse displacements, the string is kept under tension by a force T and clamped at the points $x = 0, l$. Let $y = y(x, t)$, $z = z(x, t)$ be the transverse displacements of the points of the string with the Eulerian coordinate x, $0 \leqslant x \leqslant l$. We calculate the elongation $d\Delta$ of an infinitesimal element dx accompanying its orthogonal displacements y, z from the equilibrium position $y = z = 0$. For the stretched element, when account is taken of the extensibility of the string material, we have the expression $ds = (1 + h^2)^{1/2}dx$, where $h^2 = y'^2 + z'^2$ and derivatives with respect to x are denoted by primes. The required elongation $d\Delta \equiv ds - dx = \lambda dx$ and the expression for the coefficient λ can be represented by a series which converges when $h^2 < 1$

$$\lambda = h^2\left[1+\left(1+h^2\right)^{1/2}\right]^{-1} = h^2/2 - h^4/8 + h^6/16 - \ldots \tag{1.1}$$

Actually, expansion (1.1) for λ is usually truncated after the first term. When account is taken of the extensibility, the total tension of the string T^* will depend on the point x and the time t

$$T^* = T + ESd\Delta/dx = T + \lambda ES, \quad T^* > T \tag{1.2}$$

Here, E is Young's modulus of the material and S is the area of cross-section of the string fibre. Hence, ES is the tensile stiffness and formula (1.2) reflects Hooke's law.

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We now calculate the elementary work dA for the stretching and the total potential energy U of the string due to the transverse displacements $y(x, t)$, $z(x, t)$ [5, 7].

$$dA = dA_T + dA_E, \quad dA_T = -T\lambda dx, \quad dA_E = -\tfrac{1}{2} ES\lambda^2 dx$$

$$U = -\int_0^l dA = \int_0^l P(h^2)dx \tag{1.3}$$

$$P \equiv T\lambda + \tfrac{1}{2} ES\lambda^2 = Th^2/2 + Nh^4/8 - Nh^6/16 + \ldots, \quad N = ES - T$$

Assuming that the magnitude of h^2 is sufficiently small, we can confine ourselves to terms up to $O(h^4)$ in (1.3) and discard the $O(h^6)$ terms. The coefficient $N > 0$ for elastic systems with a stiff characteristic. Moreover, in the case of metal strings (steel strings, in particular), the strong inequality $ES \gg T$ is satisfied and, usually, $ES/T \sim 10^2$. We now write down the expressions for the kinetic energy K of the transverse displacements of the elements of the string and the work W done by the external distributed forces $F^{y,z}(x, t)$

$$K = \frac{\rho}{2}\int_0^l \left(\dot{y}^2 + \dot{z}^2\right)dx, \quad W = \int_0^l \left(F^y y + F^z z\right)dx \tag{1.4}$$

Here, ρ is the linear density of the string which, for simplicity, is assumed to be constant, and differentiation with respect to time is indicated by a dot. The equations of the vibrations are obtained using the Ostrogradskii–Hamilton variational principle for the Lagrangian function $L = K - U - F^y y - F^z z$ [1, 2, 5, 8]. When account is taken of the boundary conditions and the initial distributions of the y, z-displacements and the velocities $\dot{y}$, $\dot{z}$, we obtain

$$\begin{gathered}
\rho\ddot{y} = Ty'' + N\left[\left(y'^2 + \tfrac{1}{2}h^2\right)y'' + y'z'z''\right] + F^y \\
\rho\ddot{z} = Tz'' + N\left[y'y''z' + \left(z'^2 + \tfrac{1}{2}h^2\right)z''\right] + F^z \\
y(0,t) = y(l,t) = 0, \quad z(0,t) = z(l,t) = 0 \\
y(x,0) = d^y(x), \quad \dot{y}(x,0) = g^y(x) \\
z(x,0) = d^z(x), \quad \dot{z}(x,0) = g^z(x)
\end{gathered} \tag{1.5}$$

It is difficult to investigate the non-linear initial-boundary-value problem (1.5) in an exact formulation. An approach is proposed which is based on the so-called single-mode approximation [4–8]. The solution $y(x, t)$, $z(x, t)$ is constructed in the form of the series

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin \pi n\frac{x}{l}, \quad z(x,t) = \sum_{m=1}^{\infty} b_m(t)\sin \pi m\frac{x}{l} \tag{1.6}$$

which automatically satisfy the zero boundary conditions. The system of functions $\{\sin \pi n l^{-1} x\}$ is complete $[x \in [0, l])$ and these functions will be eigenfunctions if one neglects non-linearities, and, hence, after substituting series (1.6) into (1.5), expanding the functions $F^{y,z}$ for this system and equating the Fourier coefficients of like harmonies, we obtain two coupled denumerable systems of equations for $a_n(t)$, $b_m(t)$. The coupling is due to a cubic non-linearity which is assumed to be fairly weak as a consequence of the smallness of the amplitudes of the partial vibrations. This is achieved by the choice of the distributions for $F^{y,z}(x, t)$ and $d^{y,z}(x)$, $g^{y,z}(x)$ which only contain the mode under consideration with number k ($n = m = k$). In practice, the fundamental mode of vibration $k = 1$ is usually implemented.

Thus, we set the harmonics $F_n^y(t) = F_m^z(t) \equiv 0$, $d_{n,m}^{y,z} = g_{n,m}^{y,z} = 0$ when $n, m \neq k$ and the effect of the other modes on the kth mode may then be considered to be insignificant and, instead of the series (1.6), we take representations of the form

$$y(x,t) = a_k(t)\sin \pi k l^{-1} x, \quad z(x,t) = b_k(t)\sin \pi k l^{-1} x \tag{1.7}$$

Substitution of expressions (1.7) into the Lagrangian L enables us to obtain its single-mode approximation in which a_k, b_k are generalized coordinates. The Lagrangian equations of motion are reduced to the form of a system which describes the motion of a plane oscillator with a cubic non-linearity. We now formulate the Cauchy problem

$$\begin{gathered} \ddot{a}_k + \omega_k^2 a_k + \gamma_k(a_k^2 + b_k^2)a_k = f_k^a(t) \\ \ddot{b}_k + \omega_k^2 b_k + \gamma_k(a_k^2 + b_k^2)b_k = f_k^b(t), \quad t \geqslant 0 \\ a_k(0) = d_k^y, \quad \dot{a}_k(0) = g_k^y, \quad b_k(0) = d_k^z, \quad \dot{b}_k(0) = g_k^z \\ \omega_k^2 = \left(\frac{\pi k}{l}\right)^2 \frac{T}{\rho}, \quad \gamma_k = \frac{3}{8}\left(\frac{\pi k}{l}\right)^4 \frac{N}{\rho}, \quad f_k^{a,b}(t) = \frac{F_k^{y,z}(t)}{\rho} \end{gathered} \tag{1.8}$$

Equations (1.8) and the corresponding initial conditions can also be derived from the initial-boundary-value problem (1.5) using the Fourier method. We note that, when the Kirchhoff hypothesis is satisfied [4, 6, 8], the relation between the different modes of vibration will be parametric, that is, for each mode the cubic non-linearity leads to the expressions $\Sigma_k(a_n, b_m)a_k$, $\Sigma_k(b_m, a_n)b_k$, where Σ_k are series of all a_n, b_m. The zero initial conditions and the absence of excitation of the corresponding modes will lead to the trivial solutions, $a_n \equiv 0, b_m \equiv 0$. Moreover, Kirchhoff's hypothesis leads to constancy of the tension throughout the length of the string (but not in time).

A system with two degrees of freedom (1.8) is next investigated with various assumptions regarding the parameters, the initial conditions and the external disturbance. The theoretical and applied aspects of the formulation of problems of the free vibrations ($f_k^{a,b} \equiv 0$) and the steady forced vibrations caused by a harmonic excitation ($f_k = f \cos \Omega t$), that is, the so-called resonance or amplitude-frequency characteristics, are of interest. Since the value of k is fixed (to be specific, the fundamental mode of vibration $k = 1$ is considered), the subscript k is omitted for brevity and the more convenient notation $a^0, \dot{a}^0, b^0, \dot{b}^0$ is adopted for the initial values.

Together with the issue of the existence and the construction of the solutions, the investigation of their stability proves to be important in practice. We note that system (1.8) can be represented in canonical Hamiltonian form ($\dot{a} = p_a, \dot{b} = p_b$ are momenta).

2. ANALYSIS OF THE FREE VIBRATIONS OF A STRING

We will now consider the Cauchy problem (1.8) when $f^{a,b}(t) \equiv 0$ and introduce the dimensionless time t^*, normalized variables a^*, b^* and the parameter ε as follows (the asterisk is henceforth omitted):

$$\begin{gathered} t^* = \omega t, \quad a^* = a d_0^{-1}, \quad b^* = b d_0^{-1}, \quad \varepsilon = \gamma\omega^{-2} d_0^2 \\ \ddot{a} + a + \varepsilon(a^2 + b^2)a = 0, \quad \ddot{b} + b + \varepsilon(a^2 + b^2)b = 0 \end{gathered} \tag{2.1}$$

Here, d_0 is the scale of the change in a and b for which the cubic terms turn out to be small and the quantities $a^*, b^* \sim 1$. This scale is specified by the choice of the initial quantities $a^0, \dot{a}^0, b^0, \dot{b}^0$.

We note that system (2.1) has both the solution $a(t) \equiv 0$ and the solution $b(t) \equiv 0$. System (2.1) can be integrated for arbitrary values of the parameter ε. It admits of two elementary integrals, an "energy" integral E and an "angular momentum" integral C (an "area integral"). Further integration leads to elliptic functions. It is more convenient to carry out the analytical integration procedure in polar coordinates r, φ by means of the substitution $a = r \sin \varphi$, $b = r \cos \varphi$. As a result, we obtain

$$\begin{gathered} E = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2(1 + \dot{\varphi}^2) + \tfrac{1}{4}\varepsilon r^4, \quad C = r^2\dot{\varphi} \\ t - t_0 = \pm\int_{r^0}^{r} \frac{d\xi}{\upsilon(\xi)}, \quad \varphi - \varphi^0 = \pm C\int_{r^0}^{r} \frac{d\xi}{\xi^2\upsilon(\xi)} \\ \upsilon(r) = r^{-1}(2Er^2 - r^4 - \tfrac{1}{2}\varepsilon r^6 - C^2)^{1/2} \end{gathered} \tag{2.2}$$

The range of the change in the radius vector r is determined by the roots of the equation $\upsilon(r) = 0$ and, in the general case, there are two such quantities $r_{\min}(E, C)$ and $r_{\max}(E, C)$. There are two degenerate modes of motion. The vibrations along the straight line ($C = 0$, that is, $\varphi \equiv 0$), which intersects the origin of coordinates, occur within the limits $r_{\min}(E, 0) = 0$, $r_{\max}(E, 0) = \varepsilon^{-1/2}[(1 + 4\varepsilon E)^{1/2} - 1]^{1/2}$. In the case of motion along a circle $r_0 = r_{\min}(E, C) = r_{\max}(E, C)$, the value of r_0 is a root of the equation $r^4 + \varepsilon r^6 = C^2$ subject to the condition $2E = r_0^2 + C^2 r_0^{-2} + 1/2\varepsilon r_0^4$ and, as a result

$$r_0 = r_0(E) = (2/(3\varepsilon))^{1/2}\left[(1+3\varepsilon E)^{1/2} - 1\right]^{1/2}, \quad C = C_0(E) = \pm r_0^2(1+\varepsilon r_0^2)^{1/2}$$

A complete analysis of the motion on the basis of expressions (2.2) requires extremely tedious calculations which are unjustified in the case of the small values of ε being considered here. It is more constructive to present an asymptotic analysis of the quasilinear oscillations of system (2.1) using the method of averaging [10, 11]. The change to a system in the standard form can be achieved by several methods such as, for example, by introducing new "amplitude-phase" variables, an "amplitude-phase detuning" or by changing to osculating variables of the Van der Pol type [10], and so on. Each of these substitutions has certain advantages and disadvantages. The choice must be dictated by the mechanical content and by clarity. In the case of systems of the type (2.1), it is preferable to take the slow variables of "amplitude-phase detuning" [7]

$$a = A\cos(t+\alpha), \quad b = B\cos(t+\beta), \quad \dot a = \partial a/\partial t, \quad \dot b = \partial b/\partial t \tag{2.3}$$

Here, A and B are amplitudes, and α and β are the phase corrections of the partial oscillations. In the first approximation of the method of averaging, one obtains the equations

$$\begin{gathered} \dot A = -AB^2\sin 2\delta, \quad \dot B = A^2 B\sin 2\delta, \quad \delta = \beta - \alpha \\ \dot\alpha = 3A^2 + 2B^2 + B^2\cos 2\delta, \quad \dot\beta = 2A^2 + A^2\cos 2\delta + 3B^2 \end{gathered} \tag{2.4}$$

for A, B, α, β.

In (2.4) and subsequently, derivatives with respect to the "slow time" $\tau = \varepsilon t/8$, $\tau \sim 1$ are denoted by dots. The initial values of the variables are determined using (2.3).

The integrals (2.2) take the form

$$A^2 + B^2 = 2E, \quad AB\sin\delta = -C$$

It follows from the equation for δ which is obtained from (2.4): $\dot\delta = -2(A^2 - B^2)\sin^2\delta$, that motion along a line (in "phase" and in "antiphase") corresponds to the stationary points $\delta^* = 0, \pi$. The value $\delta = \pm\pi/2 (C = \mp E)$ corresponds to motion along a circle $A = B = E^{1/2}$. Note that the quantities being considered differ from the exact quantities by $O(\varepsilon)$ for $t \sim 1/\varepsilon$. $B(\tau) = \text{const}$ corresponds to the particular solution $A(\tau) \equiv 0$ of system (2.1) whereas $-B(\tau) \equiv 0$ corresponds to $A(\tau) = \text{const}$.

Expressions for A, B and δ are found using elementary functions [7]

$$\begin{gathered} A^2, B^2 = E \pm D\cos(4C\tau + \theta), \quad \sin\delta = -C(AB)^{-1} \\ D = (E^2 - C^2)^{1/2}, \quad 0 \leqslant D \leqslant E, \quad \theta = \text{const} \end{gathered} \tag{2.5}$$

For motion along a circle, $D = 0$, that is, $E = |C|$ while, in the case of vibrations along a straight line, $D = E$ ($C = 0$) and, then, $\delta = \pm\pi/2$ or $\delta = 0$, respectively. It also follows from (2.5) that the amplitudes A and B vary within the limits from $(E - D)^{1/2}$ to $(E + D)^{1/2}$. Using expressions (2.5) which have been found, for $\dot\alpha$, $\dot\beta$ we obtain explicit representations in terms of τ which are integrated in terms of elementary functions

$$\begin{gathered} \dot\alpha = 6E - 2C^2A^{-2}, \quad \dot\beta = 6E - 2C^2B^{-2} \\ \alpha, \beta = \alpha^0, \beta^0 + 6E\tau - \operatorname{Arctg}\left[(E \mp D)C^{-1}\operatorname{tg}(2C\tau + \theta/2)\right] + \operatorname{arctg}\left[(E \mp D)C^{-1}\operatorname{tg}\theta/2\right] \\ \langle\dot\alpha\rangle = \langle\dot\beta\rangle = 6E - 2|C|, \quad \alpha^0 = \alpha(0), \quad \beta^0 = \beta(0) \end{gathered} \tag{2.6}$$

Thus, the approximate expressions for the variables a and b are constructed in accordance with (2.3), (2.5) and (2.6). The main properties of the motion have been found. We also mention a rather important fact. It follows from (2.5) that plane motions ($C = 0$) are unstable in the following sense. The presence of a $|C| \gg \varepsilon > 0$ which may be as small as desired, will lead to a state where, during the process of evolution, the "plane of vibrations" will turn and, according to (2.5), the string will perform rapid oscillations in an arbitrary plane. The dimensionless time for the rotation of the plane by an angle of $\pi/2$, that corresponds to the rotation of the phase by an angle of π, has a magnitude $\Delta t = 2\pi(\varepsilon C)^{-1}$. For small $|C|$, the trajectory a, b of system (2.1), apart from a quantity $\varepsilon \ll |C|$, is a very prolate ellipse with major and minor axes $(E + D)^{1/2}$ and $(E - D)^{1/2}$, respectively

$$(E+D)^{1/2} = (2E)^{1/2}\left(1+\kappa^2/8+O(\kappa^4)\right), \quad \kappa^2 = C^2E^{-2}$$

$$(E-D)^{1/2} = (E/2)^{1/2}|\kappa|\left(1-\kappa^2/8+O(\kappa^4)\right), \quad |\kappa| \ll 1$$

It can be shown that, a for a fixed τ, relations (2.3) and (2.5) define a rotated ellipse with constant semi-axes $(E + D)^{1/2}$ and $(E - D)^{1/2}$. The axes of the ellipse slowly rotate at a velocity $\varepsilon C/4$. These properties follow directly from expressions (2.3) and (2.5) after the fast time t has been eliminated and they have been reduced to the form

$$a^2A^{-2} + b^2B^{-2} - 2aA^{-1}bB^{-1}\cos\delta = \sin^2\delta \tag{2.7}$$

In spite of the fact that the parameters A, B and δ in (2.7) are functions of τ, the semi-axes turn out to be constant and equal to the values shown above in the rotating system of coordinates which corresponds to the canonical form of the ellipse.

Hence, motions with different velocity scales are observed in the system. In the first place, these are rapid vibratory motions with a velocity $O(1)$. Secondly, when $|C| \sim 1$ $E \sim 1$, an evolution of the axes of the ellipse occurs with a velocity $O(\varepsilon)$. If $\varepsilon \leqslant |C| \leqslant E \sim 1$, then there are three scales: with velocities of $O(1)$, $O(|C|)$ and $O(\varepsilon|C|)$. It is interesting to note that the velocity of rotation of the plane in the slow time τ is equal to $2C$ (rather than $4C$, see (2.5)).

Experimental observations of the free vibrations of a string require an extremely high precision and selectivity as well as a high accuracy in specifying the initial values. The presence of perturbing factors under real conditions leads to the need to employ "resonance methods" which are less subject to the effect of the above-mentioned perturbations [4–6].

3. CONSTRUCTION AND ANALYSIS OF THE FORCED STEADY VIBRATIONS OF A STRING

3.1. *Preliminary transformations*

Interesting properties of the spatial and plane vibrations of a string are also revealed in the case of a periodic excitation. Let an external disturbance occur in a certain fixed plane which passes through the line of the undeformed string. Without any loss of generality, we shall assume that $f^a(t) \not\equiv 0, f^b(t) \equiv 0$ and consider a harmonic exitation $f^a(t) = f\cos\Omega t$. As in Section 2, we introduce dimensionless variables and parameters as follows (the subscript k is omitted)

$$t^* = \omega t, \quad a^* = ad_0^{-1}, \quad b^* = bd_0^{-1}, \quad \mu = \Omega\omega^{-1} \tag{3.1}$$

$$\varepsilon\beta = \gamma\omega^{-2}d_0^2, \quad \beta \sim 1, \quad \varepsilon = f\omega^{-2}d_0^{-1}, \quad 0 < \varepsilon \ll 1$$

Henceforth, the asterisk is omitted for brevity.

As a result of the transformations (3.1), a quasilinear vibratory system with weak coupling between the subsystems is obtained from (1.8)

$$\ddot{a} + a + \varepsilon\beta(a^2+b^2)a = \varepsilon\cos\mu t$$

$$\ddot{b} + b + \varepsilon\beta(a^2+b^2)b = 0 \tag{3.2}$$

The problem of constructing and investigating the steady-state forced vibrations of system (3.2) in the neighbourhood of the main resonance $\mu \simeq 1$ is formulated. Here, the initial value of the variables are not specified. They are determined when solving the problem and can be chosen from a certain sufficiently small neighbourhood of these values [9–11].

The problem of the existence, construction and investigation of the stability of the periodic motions of system (3.2) can be solved using the well-developed and well-founded Lyapunov–Poincaré methods [9]. In order to investigate the dependence of the amplitudes of the partial vibrations on the frequency it is more convenient to change to variables of the "amplitude-phase detuning" type, analogous to (2.3), using the formulae

$$\begin{aligned} a &= A\cos(\mu t+\theta), \quad \dot a = \partial a/\partial t \\ b &= B\cos(\mu t+\psi), \quad \dot b = \partial b/\partial t \end{aligned} \tag{3.3}$$

Here, we shall assume that the (relative) frequency μ of the external disturbance lies in an ε-neighbourhood of the main resonance $\mu = 1 + \varepsilon\lambda$, where $\lambda \sim 1$. By virtue of the perturbation of system (3.2), the variables A, B, θ, ψ will then be slow. System (3.2) is reduced to the standard Bogolyubov form [10, 11]: $\dot x = \varepsilon X(\mu t, x)$, where x is the vector of the indicated osculating variables and X is a periodic function of t which is smooth when $A > 0$. In the case of a standard system, the issue of the existence and the approximate construction of a periodic solution $x = x(\mu t, \varepsilon)$ reduces to investigating the existence and uniqueness (non-degeneracy) of the stationary point ξ^* which corresponds to the averaged system $\dot\xi = X_0(\xi)$, where $X_0(x)$ is the mean of $X(\mu t, x)$ with respect to t. The sufficient conditions reduce to the following relations

$$\xi^* = \arg X_0(\xi), \quad \det X_0'(\xi^*) \neq 0 \tag{3.4}$$

The desired periodic solution x then has the form $x(\mu t, \varepsilon) = \xi^* + \varepsilon\varphi(\mu t, \varepsilon)$, where the periodic function φ is constructed by quadratures from known functions using the method of successive approximations [9, 10].

In the case of the variables A, B, θ, ψ, the averaged system in the slow time $\tau = \varepsilon t$ has the form (for averaged variables the previous notation is retained)

$$\begin{aligned} \dot A &= -\tfrac12\sin\theta - (\beta/8)AB^2\sin 2\delta \\ \dot B &= (\beta/8)A^2B\sin 2\delta, \quad \delta = \psi - \theta \\ \dot\theta &= -\lambda - \tfrac12 A^{-1}\cos\theta + (3\beta/8)A^2 + (\beta/4)B^2(1+\tfrac12\cos 2\delta) \\ \dot\psi &= -\lambda + (\beta/4)A^2(1+\tfrac12\cos 2\delta) + (3\beta/8)B^2 \end{aligned} \tag{3.5}$$

Note that system (3.5) does not have a Hamiltonian form. However, a transformation from the variables A, B, θ, ψ to variables of the Van der Pol type enables one to obtain an averaged Hamiltonian system. This is equivalent to replacing a, $\dot a$, b, $\dot b$ by the Van der Pol variables in the initial system (3.2) and subsequent averaging with respect to t (see (4.2)).

System (3.5) allows of a set of stationary points which correspond to the different modes of vibration. We shall now consider them.

3.2. *Plane steady vibrations*

These are defined by the relations (see (3.4))

$$\lambda = \mp\tfrac12 A^{-1} + (3\beta/8)A^2, \quad A>0, \quad B=0, \quad \theta = 0, \ \pi \tag{3.6}$$

The amplitude of the vertical vibrations $A_0 = A_0^{\mp}(\lambda, \beta)$ is found as the root of a cubic equation. An analytical solution is an extremely time-consuming problem. The graphical construction of the family of functions $A_0^{\mp}(\lambda, \beta)$ is quite elementary if the inverse function $\lambda(A, \beta)$ is constructed according to (3.6). Such a solution when $\beta = 1$ is shown in Fig. 1. The unique branch $A_0^-(\lambda, \beta)$, which is defined for all $\lambda \in (-\infty, \infty)$ where $\beta > 0$ is a parameter, corresponds to "in-phase" vibrations ($\theta = 0$), that is, the minus sign in (3.6). This branch increases monotonically with respect to λ. When $\lambda = 0$, the value $A_0^-(0, \beta) = A_0^* = (4/(3\beta))^{1/3}$ is obtained for A_0^-. When $|\lambda| \ll 1$, the expansion $A_0^- = A_0^* + (2\lambda/3)A_0^{*2} + O(|\lambda|^3)$ holds for $A_0^-(\lambda, \beta)$, that is, $\lambda = 0$ is a point of inflection (see Section 3.3). The asymptotic behaviour is

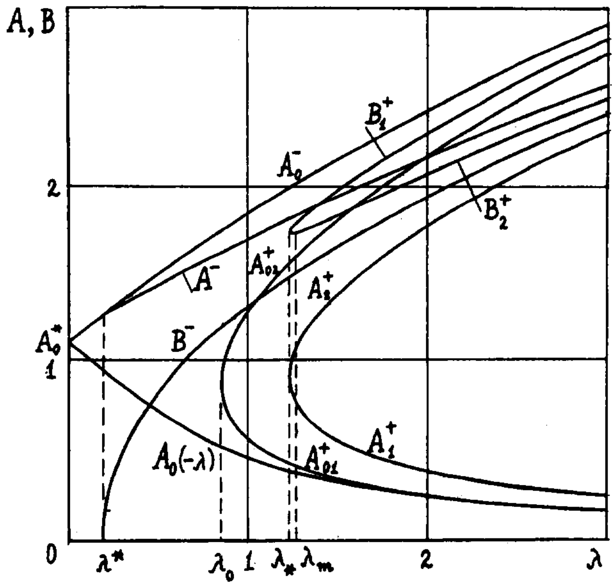

Fig. 1.

described by the expressions: $A_0^-(\lambda, \beta) = -(2\lambda)^{-1} - (3\beta/4)(2\lambda)^{-4} + O(|\lambda|^{-7})$ when $\lambda \to -\infty$. For a positive frequency detuning we have $A_0^-(\lambda, \beta) = (8\lambda/(3\beta))^{1/2} + (4\lambda)^{-1} + O(\lambda^{-2})$ when $\lambda \to +\infty$. In the expressions which have been presented the parameter $\beta \sim 1$ ($\beta > 0$). The case when $\beta \to 0$ leads to a linear oscillator: $A_0^-(\lambda, \beta) = -(2\lambda)^{-1} - (3\beta/4)(2\lambda)^{-4} + O(\beta^2)$, where $\lambda < 0$ and, when $\beta \to \infty$ for $\lambda \sim 1$, we obtain the asymptotic form $A_0^-(\lambda, \beta) = (4/3\beta)^{1/3} + (2\lambda/3(4/(3\beta))^{2/3} + O(\beta^{-1})$.

We will now consider the "antiphase" vibrations ($\theta = \pi$) to which the plus sign in relation (3.6) corresponds. The curve $A_0^+(\lambda, \beta)$ is defined for $\lambda \geqslant \lambda_0 = (3/4)(3\beta/2)^{1/3}$ and, moreover, $A_0^+(\lambda, \beta) = (2/(3\beta))^{1/3}$. When $\lambda > \lambda_0$, it has two branches: a falling branch A_{01}^+ and a rising branch A_{02}^+ (when $\lambda \to \infty$). The two curves join smoothly at the point $\lambda = \lambda_0$ at which the tangent to the line is vertical. The asymptotic behaviour of the following curve A_{01}^+ as $\lambda \to \infty$ is analogous to the behaviour of A_0^- as $\lambda \to -\infty$, that is, we have $A_{01}^+ = (2\lambda)^{-1} + (3\beta/4)(2\lambda)^{-4} + O(\lambda^{-7})$. In a similar manner, for the rising branch A_{02}^+ we have the approximate expression $A_{02}^+ = (8\lambda/(3\beta))^{1/2} - (4\lambda)^{-1} + O(\lambda^{-3/2})$. It follows from the representations which have been obtained that $A_0^- - A_{02}^+ = (2\lambda)^{-1} + O(\lambda^{-3/2}) \to 0$ when $\lambda \to \infty$ and, for the falling part of the curve A_0^- and the branch A_{01}^+, we similarly obtain a faster descent $A_0^-(-\lambda, \beta) - A_{01}^+(\lambda, \beta) = -(3\beta/2)(2\lambda)^{-4} + O(\lambda^{-7})$ as $\lambda \to \infty$.

We will now consider the asymptotic form of A_0^+ with respect to the parameter β. As $\beta \to 0$, the asymptotic form $A_0^+ = (2\lambda)^{-1} + (3\beta/4)(2\lambda)^{-4} + O(\beta^2)$, $\lambda > 0$ holds. If $\beta \to \infty$, then the values $\lambda \geqslant \lambda_0 = (3/4)(3\beta/2)^{1/3} \to \infty$ and the expansions for $A^+{}_0$ will therefore depend on the estimates $\lambda = \lambda(\beta)$, in a class of power series, for example. Let $\lambda = \Lambda\beta^{1/3}$, where the constant $\Lambda > (3/4)(3/2)^{1/3}$. Then, the estimate $A_0^+ = \zeta^* \beta^{-1/3}$ where ζ^* is the positive root of the equation $\zeta^3 - (8/3)\Lambda\zeta - 4/3 = 0$. If $\lambda = \Lambda\beta^{\nu}$, $1/3 < \nu < 1$, then the approximate expression $A_0^+ = \zeta\beta^{-1/3}$, where $\zeta = 4/(3\chi) + (4/(3\chi))^3 + O(\chi^{-5})$ and the magnitude of $\chi = (8/3)\Lambda\beta^{\nu - 1/3} \to \infty$ as $\beta \to \infty$. Let $\nu = 1$, then we have two asymptotic forms; an asymptotic form $A_{01}^+ = (2\Lambda\beta)^{-1} + (3/(8\Lambda))(2\Lambda\beta)^{-3} + O(\beta^{-5})$ which decreases without limit and a finite asymptotic form $A_{02}^+ = \kappa + \mu a_1 + \mu^2 a_2 + O(\mu^3)$ where the magnitude of the parameter $\kappa = (8\Lambda/3)^{1/2}$ is of the order of unity with respect of $\beta \to \infty$, $\mu = (2\Lambda\beta)^{-1}$ is a small parameter and the expansion coefficients $a_1 = -1/2\kappa^{-2}$, $a_2 = -3/(8\kappa^5)$, and so on. Finally, when $\nu > 1$, the asymptotic form $A_0^+ = (2\Lambda\beta^{\nu})^{-1} + (3/64)\Lambda^{-4}\beta^{-4\nu+1} + O(\beta^{-7\nu+2})$ is obtained as $\beta \to \infty$.

In practice, the possibility of obtaining plane vibrations is determined by their stability, the investigation of which is postponed until Section 4.

3.3. *Steady spatial vibrations*

We now consider the general case when $A, B > 0$. The system of spatial vibrations is determined by the relations (see (3.4))

$$\theta = 0,\ \pi;\quad \lambda = \mp(2A)^{-1} + (3\beta/8)A^2 + (\beta/8)B^2,\ \beta > 0$$
$$\psi = \pm\pi/2;\quad \lambda = (\beta/8)A^2 + (3\beta/8)B^2,\ |\lambda| < \infty \tag{3.7}$$

The signs $\mp$ in the first equation of (3.7) for A, B correspond to the values $\theta = 0, \pi$, respectively, and do not depend on the signs of $\psi = \pm\pi/2$. When $B = 0$, this equation determines the amplitude $A_0^{\mp}(\lambda, \beta)$ investigated above. We eliminate $B^2 > 0$ from it using the second equation and obtain relations which are convenient in the subsequent analysis

$$\lambda = \mp 3(4A)^{-1} + (\beta/2)A^2,\quad B = [\mp 2(\beta A)^{-1} + A^2]^{1/2} \tag{3.8}$$

Formulae (3.8) enable us to construct a graphical solution $A(\lambda, \beta)$, $B(\lambda, \beta)$ as in Section 3.2 without determining the roots of the cubic equation in A. It follows from (3.8) that the case of spatial vibrations corresponds to a higher equivalent stiffness of the non-linear characteristic and a greater amplitude of the external disturbance (since 1/2 > 3/8 and 3/4 > 1/2, respectively, see (3.6)). The analysis of the family of curves $A^{\mp}(\lambda, \beta)$ is analogous to the analysis carried out above in the case of $A_0^{\mp}(\lambda, \beta)$. The family of curves $B^{\mp}(\lambda, \beta)$ is analogous to the analysis carried out above in the case of $A_0^{\mp}(\lambda, \beta)$. The family of curves $B^{\mp}(\lambda, \beta)$ can be investigated using the second formula of (3.8) and the asymptotic form of the expressions $A^{\mp}(\lambda, \beta)$.

We will first consider the case when $\theta = 0$ ("in-phase" vibrations) which corresponds to the minus sign in (3.8). The family of curves $A^-(\lambda, \beta)$ is a monotonically increasing function λ, $\lambda \in (-\infty, \infty)$ for a fixed $\beta > 0$. These curves intersect the ordinate axis ($\lambda = 0$), taking the values $A^0(\beta) = A^-(0, \beta) = (3/(2\beta))^{1/3}$. The expansions

$$A^-(\lambda,\beta) = -3(4\lambda)^{-1} - (\beta/(2\lambda))(3/(4\lambda))^3 + O(|\lambda|^{-7}),\ \lambda \to -\infty$$
$$A^-(\lambda,\beta) = (2\lambda/\beta)^{1/2} + 3(8\lambda)^{-1} + O(\lambda^{-2}),\ \lambda \to +\infty \tag{3.9}$$

hold for asymptotically large values of $|\lambda|$.

It is interesting to note that, for small $|\lambda|$, the approximate expression $A^-(\lambda, \beta) = A^0(\beta) + (4/9)\lambda A^{02}(\beta) + O(\lambda^3)$ holds for $A^-(\lambda, \beta)$, that is, each curve of the family has zero curvature (a point of inflection) at the point $\lambda = 0$. The case when $\beta = 0$ leads to a linear oscillator. When $\beta \ll 1$, we obtain the estimate (3.9) corresponding to $\lambda < 0$. However, the error in β will be $O(\beta^2)$ (see above, the case of plane vibrations). The asymptotic form of A^- with respect to β, as $\beta \to \infty$, $\lambda \sim 1$, is given by $A^-(\lambda, \beta) = (3/(2\beta))^{1/3} + (4\lambda/9)(3/(2\beta))^{2/3} + O(\beta^{-1})$.

We will now consider expression (3.8) for $B = B^-$ which corresponds to the minus sign ($\theta = 0$). It follows from it that vibrations along the z-axis occur at quite large values of A, that is, of the frequency detuning λ. Actually, $B^2 > 0$ if $A^- > (2/\beta)^{1/3}$ which is possible for sufficiently large λ: $\lambda > \lambda^* = (1/4)(\beta/2)^{1/3}$. It follows from (3.6) that $A_0^-(\lambda^*, \beta) = A^-(A^*, \beta)$, that is, the point $\lambda = \lambda^*$ is a critical point and vibrations corresponding to two modes are possible at this point. As λ increases ($\lambda > \lambda^*$), either plane ($B = 0$) or spatial vibrations will occur which is determined by their stability. (For the solution of this problem, see Section 4.) Spatial vibrations are therefore impossible ($B^2 < 0$) when $\lambda < \lambda^*$, and the asymptotic forms (3.9) obtained above for A^- as $\lambda \to -\infty$ and the expansion with respect to small $|\lambda|$, $\lambda < 0$ do not correspond to reality. When $\lambda > \lambda^*$ is increased, the curve $B^-(\lambda, \beta)$ grows quite rapidly and, furthermore, the derivative of B^- with respect to λ at the point $\lambda = \lambda^* + 0$ is infinite, since the asymptotic form $B^-(\lambda, \beta) \simeq (24/(11\beta))^{1/2}(\lambda - \lambda^*)^{1/2}$ holds for B^-. According to (3.8), (3.9), the approximate expression $B^-(\lambda, \beta) = (2\lambda/\beta)^{1/2} + (8\lambda)^{-1} + O(\lambda^{-2})$ holds for B^- as $\lambda \to \infty$. By analogy with the case considered above for $A_0^+(\lambda, \beta)$, the asymptotic form $B^-(\lambda, \beta)$ with respect to β, $\beta \to \infty$ is obtained in the class of power relations (see below). For values of $\beta \ll 1$, $\lambda \sim 1$, the curve of B^- is described using (3.8), (3.9) in a similar manner to A^- by the expression $B^-(\lambda, \beta) = (2\lambda/\beta)^{1/2} - (8\lambda)^{-1} + O(\beta) \to \infty$ as $\beta \to 0$. The approximate expressions for A^- and B^- show that $A^- > \beta^-$ which is confirmed by calculations (see Fig. 1).

We will now consider the behaviour of the family of resonance curves $A^+(\lambda, \beta)$, $\beta^+(\lambda, \beta)$ which are defined by relations (3.8) (with the plus sign). They correspond to spatial vibrations when ($\theta = \pi$) (in "antiphase") and $\psi = \pm\pi/2$. The curves A^+, B^+ consist of two branches $A_{1,2}^+$, $B_{1,2}^+$ which smoothly join at the cuspidal point $\lambda_* = (9/8)(4\beta/3)^{1/3} > \lambda_0 = (3/4)(3\beta/2)^{1/3}$. The corresponding values are $A_*^+ =$

$(3/(4\beta))^{1/3} > A_0^* = (2/(3\beta))^{1/3}, B_*^+$. Moreover, the tangents to these curves at the cuspidal point are vertical (as for $A_0^+(\lambda_0, \beta)$). The branches of the curve A^+ are extremely similar to the branches A_0^+ and have the asymptotic form $A_1^+ = 3(4\lambda)^{-1} + O(\lambda^{-4}), A_2^+ = (2\lambda/\beta)^{1/2} + O(1)$ as $\lambda \to \infty$. The curve of the amplitudes $B^+(\lambda, \beta)$ has a unique global minimum with respect to λ: $B_m^+(\beta) = B^+(\lambda_m, \beta) = 3^{1/2}\beta^{-1/3}(B_*^+(\beta) = (11/3)^{1/2}(3/(4\beta))^{1/3}$, where $\lambda_m = \lambda_m(\beta) = (5/4)\beta^{1/3}$. It can be shown that $\lambda_m > \lambda_*$ since $\lambda_m(\beta)/\lambda_*(\beta) = (10/9)(3/4)^{1/3} \simeq 1.013$. However, the difference is very small. Both branches of the curves $B^+ = B_{1,2}^+$ (unlike $A^+ = A_{1,2}^+$) are rising. The curve $B_1^+ = (8\lambda/(3\beta))^{1/2} + O(\lambda^{-5/2})$ corresponds to the falling branch A_1^+ and the rising branch $B_2^+ = (2\lambda/\beta)^{1/2} + O/(1)$ also corresponds to the rising branch A_2^+. It follows from the estimates which have been presented that $B_1^+ > B_2^+$, that is, the curve B_1^+, which goes higher than the curve B_2^+ corresponding to the rising branch A_2^+, corresponds to the falling branch A_1^+. A more accurate construction of the asymptotic forms $A_{1,2}^+(\lambda, \beta), B_{1,2}^+(\lambda, \beta)$ is not necessary for reasons which will be explained in Section 4.

Thus, by analysing the estimates obtained for $A_0^\mp, A^\mp, B^\mp$, a conclusion may be drawn concerning their limiting behaviour when the frequency detuning $|\lambda|$ increases. The curve A_0^- as $\lambda \to \infty$, and the branches A_{01}^+, A_1^+ for $\lambda \to \infty$ have the asymptotic form $O(|\lambda|^{-1})$. The curves $A_0^-, A_{02}^+, B_1^+ = (8\lambda/(3\beta))^{1/3}$ in the leading term of the asymptotic form and the curves $A^-, B^-, A_2^+, B_2^+ = (2\lambda/\beta)^{1/3}$, that is, they lie below the above-mentioned curves. This grouping of curves in a close neighbourhood is clearly observed in a graphical analysis of the amplitude–frequency characteristics (see Fig. 1).

In conclusion, we note that all the resonance curves are similar in the following sense. We carry out a transformation of the variables A and B and the argument λ using the formulae

$$A = U\beta^{-1/3}, \quad B = V\beta^{-1/3}, \quad \lambda = \chi\beta^{1/3}, \quad \beta > 0 \tag{3.10}$$

The equations for determining $A_0(\lambda, \beta)$ (3.6) or $A(\lambda, \beta), B(\lambda, \beta) > 0$ (3.7), (3.8) are then reduced to a form in which $\beta = 1, A = U, B = V, \lambda = \chi$ and it is therefore sufficient to construct the universal curves $U_0^\mp(\chi)$ or $U^\mp(\chi), V^\mp(\chi)$. The required characteristics for an arbitrary $\beta > 0$ are obtained according to (3.10)

$$A(\lambda,\beta) = U\left(\lambda\beta^{-1/3}\right)\beta^{-1/3}, \quad B(\lambda,\beta) = V\left(\lambda\beta^{-1/3}\right)\beta^{-1/3} \tag{3.11}$$

and, in practice, this may therefore be restricted to the construction and analysis of the curves $A(\lambda, 1)$ $B(\lambda, 1)$ which are shown in Fig. 1. The analysis performed above was carried out in accordance with tradition and for clarity.

According to (3.11), all the characteristic points as $\beta \to \infty$ move away with respect to λ and descend below the ordinate axis. On the other hand, when $\beta \to 0$, they approach the ordinate axis and increase without limit (with respect to A and B). The scale factor is equal to $\beta^{-1/3}$. Note that, on the basis of (3.10) and (3.11), it is possible to construct curves for any β if they are known for a certain fixed value.

It is of theoretical and practical interest to investigate the Lyapunov stability of the time-independent modes of vibration which have been constructed and also to model the trajectories in the neighbourhood of these motions using the exact and averaged equations of motion.

4. INVESTIGATION OF THE STABILITY OF STEADY FORCED VIBRATIONS

4.1. *Preliminary remarks*

We will investigate the stability of the steady oscillations using the first Lyapunov method [9]. To do this, we calculate the characteristic exponents of the corresponding system in variations. If only plane vibrations ($b \equiv 0$) are considered, then it is generally known [4, 9] that the steady vibrations corresponding to the resonance curve $A_{02}^+(\lambda, \beta)$ are unstable (a "saddle"-type instability). Critical points of the "centre" type correspond to the curves $A_0^-(\lambda, \beta), A_{01}^+(\lambda, \beta)$ in the linear approximation. The occurrence of a linear dissipation which may be as small as desired (of the order of ε^2, for example) leads to the asymptotic stability of these steady vibrations. The critical point for the joining of the branches A_{01}^+ and A_{02}^+ leads to a double zero characteristic exponent with a single elementary divisor which corresponds to instability. The occurrence of dissipation of the order of ε requires a separate study.

We shall now investigate the stability of steady vibrations in the linear approximation in the case of the spatial model which is described by system (3.3).

4.2. *Stability of plane steady vibrations under spatial perturbations*

The averaged equations (3.5) are not suitable for analysing the stability of plane vibrations ($B \equiv 0$) since the phase of ψ is undefined and, instead of the degenerating substitution (3.4), we therefore transform, in the case of b, $\dot b$, to osculating variables of the Van der Pol type [10, 11]

$$a = A\cos(\mu t+\theta),\ \ \dot a = -A\mu\sin(\mu t+\theta),\ \ \mu = 1+\varepsilon\lambda$$

$$b = r\cos\mu t + u\sin\mu t,\ \ \dot b = -r\mu\sin\mu t + u\mu\cos\mu t \tag{4.1}$$

On differentiating relations (4.1), by virtue of (3.3) and averaging with respect to t, we obtain, instead of (3.5) the equations for A, θ, r, u in the slow time $\tau = \varepsilon t$

$$\begin{aligned}
&\dot A = -\tfrac{1}{2}\sin\theta - (\beta/8)AB^2\sin 2\delta,\ \ B^2 = r^2+u^2\\
&\dot\theta = -\lambda - (2A)^{-1}\cos\theta + (3\beta/8)A^2 + (\beta/4)B^2(1+\tfrac{1}{2}\cos 2\delta)\\
&\dot r = -\lambda u + (\beta/8)A^2\left[-r\sin 2\theta + u(1+2\sin^2\theta)\right] + (3\beta/8)uB^2\\
&\dot u = \lambda r - (\beta/8)A^2\left[r\left(1+2\cos^2\theta\right) - u\sin 2\theta\right] - (3\beta/8)rB^2\\
&\delta = \psi-\theta,\ \ B\cos\psi = r,\ \ B\sin\psi = -u
\end{aligned} \tag{4.2}$$

The steady-state solution of system (4.2) which corresponds to plane vibrations, has the form $\theta = 0$, π; $A_0(\lambda,\beta) = A_0^-(\lambda,\beta)$, $A_{01,2}^+(\lambda,\beta)$ and the variables $r = u = 0$. The characteristic equation for the corresponding system in variations is biquadratic with respect to the exponents of p and, moreover, it has the form

$$\Delta^{\mp}(p) = Q_A^{\mp}(p^2)Q_B^{\mp}(p^2) = 0,\ \ \xi \equiv \beta A_0^{\mp}(\lambda,\beta) \tag{4.3}$$

$$Q_A^{\mp}(p^2) = p^2 + (8A_0^2)^{-1}(2\pm 2\xi),\ \ Q_B^{\mp}(p^2) = p^2 + (8A_0^2)^{-1}(2\mp\xi)$$

It follows from (4.3) that steady plane ("in phase") vibrations ($\theta = 0$) lose stability ("saddle") when $\xi > 2$. When $\xi < 2$, the characteristic exponents are pure imaginary ("centre"). The corresponding value of $\lambda(\beta)$ is calculated using formula (3.6) (with the minus sign): $\lambda(\beta) = \lambda^*(\beta) = 1/4(\beta/2)^{1/3}$. We now consider the fact that this value of λ is critical: when $\lambda > \lambda^*$, the possibility of spatial vibrations ($B > 0$) appears (see Section 3.3). Furthermore, according to (4.3), "antiphase" vibrations ($\theta = \pi$) lose stability when $\xi \geqslant 2/3$, that is, $A_0^+(\lambda,\beta) \geqslant (2/3\beta)^{1/3}$. Such are the values of the points of the curve A_{02}^+: $A_{02}^+(\lambda, \beta) \geqslant (2/3\beta)^{1/3}$. The values $A^+{}_{01}(\lambda,\beta)$ satisfy the converse inequality: $A_{01}^+(\lambda,\beta) < (2/3\beta)^{1/3}(\lambda > \lambda_0(\beta))$. These curves exist when $\lambda \geqslant \lambda_0(\beta) = (3/4)(3\beta/2)^{1/3}$ and $\lambda_0 > \lambda^*$ and the stability condition is satisfied for them. Thus, the domains of stability (in the linear approximation) and the instabilities of the steady plane vibrations of a string in the whole of the domain $|\lambda| < \infty$, $\beta > 0$ with respect to spatial perturbations have been completely determined.

4.3. *Stability of spatial steady-state vibrations*

The characteristic equation for the system in variations which corresponds to the stationary points (see (3.7)) of system (3.5) is also of biquadratic form

$$\Delta^{\mp}(p) = s^2 + ks + h = 0,\ \ \ s = p^2 \tag{4.4}$$

$$\begin{aligned}
&k = k^{\mp} \equiv (4A^2)^{-1} \pm (3\beta/8)A \pm (\beta/8)A^{-1}B^2 +\\
&+(\beta^2/4)A^2B^2,\ \ h = h^{\mp} \equiv \left[(3/64)A^{-2} \pm (\beta/16)A\right]\beta^2A^2B^2
\end{aligned}$$

The coefficients k and h are defined according to the identity in (4.4). As previously, the upper sign in $\Delta^{\mp}$ corresponds to $\theta = 0$ and the lower sign to $\theta = \pi$. In order that the roots $s_{1,2}$ of the quadratic equation should be real and negative, it is necessary and sufficient that all of the inequalities $k > 0$, $h > 0$, $d = k^2 - 4h > 0$ should be satisfied.

We now consider the "in-phase" vibrations and, for the coefficients k^-, h^-, d^-, we obtain the expressions

$$k^- = (\beta^2/4)A^4 > 0,\quad h^- = (1/64)(3+4\xi)\beta^2 B^2 > 0$$

$$d^- = \xi(2A)^{-4}(\xi^3 - 4\xi^2 + 5\xi + 6) > 0,\quad \xi > 2 \tag{4.5}$$

$$A = A^-(\lambda,\beta),\quad B = B^-(\lambda,\beta),\quad \xi = \beta A^3$$

The inequality $d^- > 0$ when $\xi > 2$ follows immediately if we make the substitution $\xi = 2 + \zeta$, $\zeta > 0$ and a third-order polynomial with positive coefficients is obtained for ζ. So, the steady spatial "in-phase" vibrations are always stable in the first approximation. The presence of a relatively small linear dissipation will lead to asymptotic stability. If the dissipation is of the order of ε, then the analysis performed above needs to be corrected to take account of this perturbation [6, 8, 9].

We will now investigate the roots of Eq. (4.4) with the plus sign ($\Delta^+(p) = 0$) which corresponds to $\theta = \pi$. For the coefficients k^+, h^+, d^+, we obtain the expressions

$$k^+ = (\beta^2/4)A^4 > 0,\quad h^+ = (64A^2)^{-1}(3-4\xi)\beta^2 A^2 B^2$$

$$d^+ = \xi(2A)^{-4}(\xi^3 + 4\xi^2 + 5\xi - 6),\quad \xi > 0 \tag{4.6}$$

$$A = A_{1,2}^+(\lambda,\beta),\quad B = B_{1,2}^+(\lambda,\beta),\quad \xi = \beta A^3$$

It immediately follows from the expression for h^+ that vibrations with amplitudes A_2^+, B_2^+, for which $\xi > \xi_* = 3/4$ are exponentially unstable (a "saddle") since $h_2^+ < 0$. The branches A_1^+, B_1^+ for which $\xi < \xi_*$, lead to the inequality $h_1^+ > 0$. For the stability of vibrations with amplitudes A_1^+, B_1^+ in the first approximation, it is sufficient that the cubic polynomial in ξ in (4.6) should be positive when $\xi < \xi_*$, that is, $\varphi_1^+(\xi) = \xi^3 + 4\xi^2 + 5\xi - 6 > 0$. By the direct substitution $\xi = \xi_* = 3/4$, we can show that $\varphi_1^+(3/4) = 27/64 > 0$. However, a root of the equation $\varphi_1^+(\xi) = 0$, which is approximately equal to $\xi_1^+ \simeq 0.72$, exists close to this value and $\varphi_1^+(\xi) < 0$ when $\xi < \xi_1^+$. Hence, the stability of the steady "antiphase" vibrations holds in the extremely narrow domain $0.75 < \xi \leqslant 0.72$. This domain is represented in the following manner in terms of the frequency detuning λ. The cuspidal point $\lambda_* = (9/8)(4\beta/3)^{1/3}$ corresponds to the value $\xi_* = 3/4$. According to (3.8), for the value of $\xi = \xi_1^+$, we obtain $\lambda_1^+ = (\beta/\xi_1^+)^{1/3}(\xi_* + \xi_1^+/2) > \lambda_*$. This domain with respect to λ is practically insignificant since the curve $A_1^+(\lambda, \beta)$ is almost vertical in a small half-neighbourhood of the cuspidal point λ_*.

Hence, a picture of the plane and spatial non-linear vibrations of a string has been completely constructed in terms of the parameters of the problem λ and β, and their Lyapunov stability has been investigated. Comparison with experimental data confirms the qualitative agreement of the results.

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